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SIMPLIFIED ALGORITHM FOR NUMERICAL SOLUTION OF LIQUID FLOW EQUATIONS

V. A. Kuznetsov UDC 519.63

An algorithm is suggested for numerical solution of differential equations for velocity and pressure on a staggered grid. The algorithm ensures unconditional convergence of iterations for a correction to pressure and without it.

Keywords: *mathematical model, incompressible liquid, velocity, pressure, correction to pressure, computational algorithm.*

Introduction. An algorithm for finding the velocity fields of an incompressible liquid (or a gas without account for its compressibility) is a constituent part of numerous mathematical models. It is based on the mass and momentum conservation laws transformed into differential equations. With gravity force ignored, the equations for stationary motion of a medium can be presented in tensor notation as follows:

flow continuity equation

$$\frac{\partial \rho u_i}{\partial x_i} = 0 \,, \tag{1}$$

Navier–Stokes equations for three components of velocity u_i

$$\frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\rho \nu \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} \,. \tag{2}$$

Here, the terms of the equations are summed up over the repeating indices i and j that number the coordinate axes x, y, and z, along which the corresponding velocity vector components u, r, and w are directed.

The equation for calculating the static pressure is obtained by artificially introducing its values into continuity equation (1). As a result, in numerical implementation of the algorithm, the problem of correctly determining the fields of pressure p and of the three components of the vector of velocity u, v, and w in an incompressible viscous fluid arises. The projection method, in which first the velocity field is constructed that does not satisfy the continuity equation and then its correction is made, can be considered a palliative solution [1]. An obligatory part of this method is an approximate discrete equation for correcting pressure whose sole function is to provide for the convergence of iterations to the desired solution of the problem [2]. According to [3], the solution of the equation for the pressure takes the greatest part of time in numerical simulation of incompressible liquid motion. In the majority of cases, to calculate the correction to the pressure, an internal iterational cycle is created, and strong under-relaxation is applied, for example, as was done in [4].

The paper presents a simplified algorithm of numerical simulation of incompressible medium motion that does not require the solution of the equations for the correction to pressure or the relaxation of this correction.

Discrete Equations in the Model of Motion of Incompressible Medium. A correct transition from differential equations to their discrete algebraic analogs is carried out on a staggered grid [2] in which control volumes for each velocity component are displaced half-step along the corresponding coordinate axis relative to the control volume for pressure. Transition to a computational control volume is accompanied by the displacement of all the symbols of the grid nodes, as shown in Fig. 1, where the control volume with the central node P for the longitudinal velocity u directed along the x axis is set out by bold lines. The central nodes of the neighboring control volumes are denoted by W, E, S, and N, and the remaining

V. G. Shukhov Belgorod State Technological University, 46 Kostyukov Str., Belgorod, 308012, Russia; email: kousnezov@mail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 91, No. 3, pp. 694–700, May–June, 2018. Original article submitted October 23, 2017.

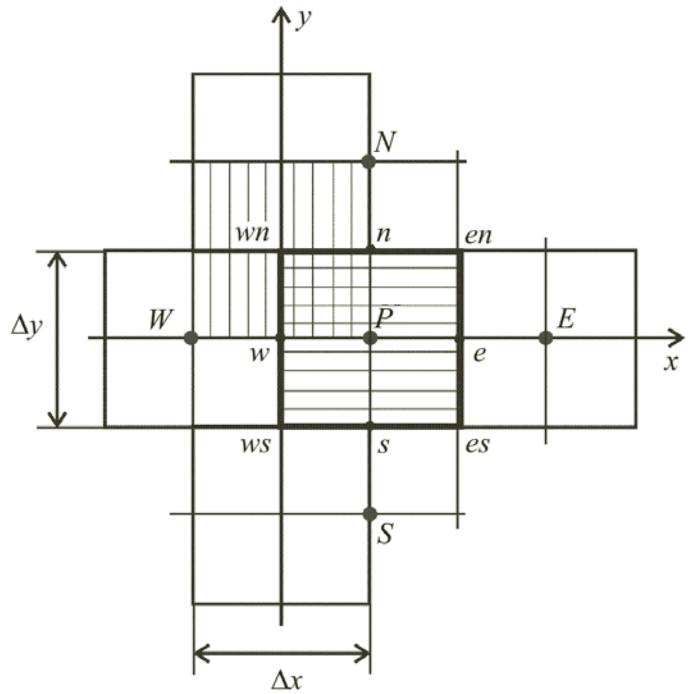

Fig. 1. Scheme of displacement of control volumes (hatched) for velocity components u and v along the x and y axes.

nodes correspond to the faces and corners of the set-out control volume. The nodes B, T, b, and t along the z axis and the corner points wb, wt, eb, and et are not shown in Fig. 1.

Integrating the Navier–Stokes equations over the displaced control volumes, we obtain a system of discrete equations for the velocity components u_i:

$$a_P^{(u_i)} u_{iP} = a_W u_{iW} + a_E u_{iE} + a_S u_{iS} + a_N u_{iN} + a_B u_{iB} + a_T u_{iT} - \Delta p_i / \Delta x_i \ , \tag{3}$$

where a_P, a_W, a_E, a_S, a_N, a_B, and a_T are the coefficients of the discrete equations; Δp_i is the difference between the pressure values over the segment Δx_i on the ith coordinate axis. The coefficient on the left hand side of equality (3) is furnished with a superscript pointing to the fact that it belongs to a definite velocity component and simultaneously to the method of its calculation as a sum of transfer coefficients that were determined on the faces of the computational control volume:

$$a_P^{(u_i)} = a_W + a_E + a_S + a_N + a_B + a_T \ .$$

If the velocity field satisfies the continuity equation (1), the substitution of the expressions for velocity components into the discrete analog of the differential continuity equation (1) leads to a discrete equation defining the pressure p_P at the central node P of the control volume [2]:

$$A_P^{(p)} p_P = A_W p_W + A_E p_E + A_S p_S + A_N p_N + A_B p_B + A_T p_T + D_P \ . \tag{4}$$

Here A_P, A_W, A_E, A_S, A_N, A_B, and A_T are coefficients, and D_P is the free term of the discrete equation for pressure calculated from the pseudovelocity values [2]. The subscript P designates the central node of the control volume for pressure. The coefficient at the sought pressure p_P also has a superscript to show its equality to the sum of the coefficients determined at the faces of this control volume:

$$A_P^{(p)} = A_W + A_E + A_S + A_N + A_B + A_T \ .$$

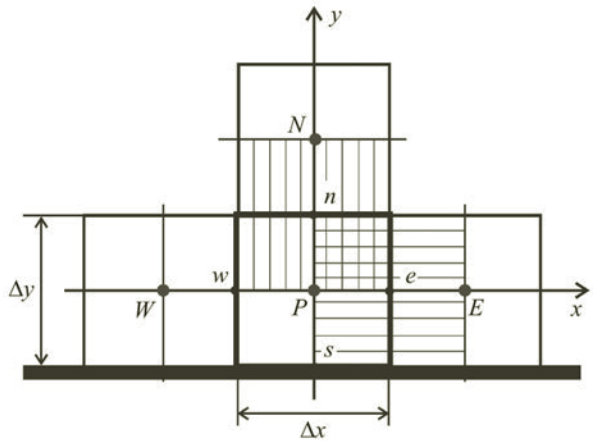

Fig. 2. Control volumes at the boundary with an immobile wall.

In the process of iterations, the velocity components usually have values that do not satisfy the continuity equation (1), as a result of which the pressure field calculated by Eq. (4) turns out to be incorrect. So that the calculated pressure field can approach the true one in the process of iterations and so that the velocity field can better satisfy the continuity equation, a correction to pressure p' is introduced. It is calculated from the approximate discrete equation analogous to Eq. (4) but with other free equation term D'_P, equal to the calculated source of mass in the control volume for pressure taken with a reversed sign [2]:

$$A_P^{(p)} p'_P = A_W p'_W + A_E p'_E + A_S p'_S + A_N p'_N + A_B p'_B + A_T p'_T + D'_P , \tag{5}$$

$$D'_P = (\rho_w u_w - \rho_e u_e)/\Delta x + (\rho_s v_s - \rho_n v_n)/\Delta y + (\rho_b w_b - \rho_t w_t)/\Delta z .$$

It was suggested to exclude the rather complex calculation of pressure on the bounding walls from the computational algorithm by applying such a system of constructing a grid, in which one of the faces of each near-wall control volume for pressure is located on the wall surface [2]. In Fig. 2, for example, a control volume for pressure is set out, the lower face of which adheres to the wall located along the x axis. As a result, the pressure p_s on the wall does not enter Eq. (4) written for this volume.

For the correction p' calculated to the pressure from the discrete equation (5) it is recommended in [2] to use the same boundary conditions as for the pressure. Probably, other more rigorous boundary conditions are also possible, taking into account the fact that the error of the pressure correction tending to zero in the course of iterations ultimately does not exert its influence on the results of numerical simulation.

Transformation of Discrete Equations. If the boundary conditions do not specify the values of p' on the surface of walls, they virtually also do not influence the value of the pressure correction at the grid nodes. A system approach shows that in the case where the influence of the boundary is not transmitted inside the computational domain, the couplings between the values of p' at the inner grid nodes, determined only approximately, turn out to be insignificant and, consequently, there is no need to take them into account. With such simplification, the right-hand side of Eq. (5) will have only the free term D'_P, whereas the sought correction to pressure will become equal to the calculated source of mass taken with a reversed sign and divided by the coefficient $A_P^{(p)}$:

$$p'_P \approx -\frac{(\rho u)_e - (\rho u)_w}{A_P^{(p)}\Delta x} - \frac{(\rho v)_n - (\rho v)_s}{A_P^{(p)}\Delta y} - \frac{(\rho w)_t - (\rho w)_b}{A_P^{(p)}\Delta z} \equiv D'_P / A_P^{(p)} . \tag{6}$$

The correction to pressure presented in a simplified form allows one to introduce its expression directly into the discrete analogs (3) of the Navier–Stokes differential equations:

$$\begin{aligned} a_P^{(u)} u_P &= a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_B u_B + a_T u_T \\ &\quad - \frac{p_e - p_w}{\Delta x} - \frac{D'_e / A_e^{(p)} - D'_w / A_w^{(p)}}{\Delta x} , \end{aligned} \tag{7}$$

$$a_P^{(v)} v_P = a_W v_W + a_E v_E + a_S v_S + a_N v_N + a_B v_B + a_T v_T - \frac{p_n - p_s}{\Delta y} - \frac{D'_n/A_n^{(p)} - D'_s/A_s^{(p)}}{\Delta y}, \quad (8)$$

$$a_P^{(w)} w_P = a_W w_W + a_E w_E + a_S w_S + a_N w_N + a_B w_B + a_T w_T - \frac{p_t - p_b}{\Delta z} - \frac{D'_t/A_t^{(p)} - D'_b/A_b^{(p)}}{\Delta z}. \quad (9)$$

Using the staggered grid here, it is well to bear in mind that transition to computation of each velocity component in the displaced control volume is accompanied by the corresponding displacement of all the designations of grid nodes.

While the longitudinal velocity u_P is calculated by the discrete equation (7) at the central node P of its control volume (see Fig. 1), the pressure and its correction in this equation are determined at the nodes w and e on the faces of the same control volume. In this case, the free term of the discrete equation (5) for the correction to pressure is calculated at the control volumes with central nodes w and e from the following expanded expressions:

$$D'_w = [(\rho u)_W - (\rho u)_P]/\Delta x + [(\rho v)_{ws} - (\rho v)_{wn}]/\Delta y + [(\rho w)_{wb} - (\rho w)_{wt}]/\Delta z ,$$

$$D'_e = [(\rho u)_P - (\rho u)_E]/\Delta x + [(\rho v)_{es} - (\rho v)_{en}]/\Delta y + [(\rho w)_{eb} - (\rho w)_{et}]/\Delta z .$$

We will use these expressions instead of the D'_w and D'_e in the equation for the longitudinal velocity u_P:

$$-D'_e/A_e^{(p)} + D'_w/A_w^{(p)} = \frac{(\rho u)_E - (\rho u)_P}{A_e^{(p)}\Delta x} - \frac{(\rho u)_P - (\rho u)_W}{A_w^{(p)}\Delta x} + \frac{(\rho v)_{en} - (\rho v)_{es}}{A_e^{(p)}\Delta y} - \frac{(\rho v)_{wn} - (\rho v)_{ws}}{A_w^{(p)}\Delta y} + \frac{(\rho w)_{et} - (\rho w)_{eb}}{A_e^{(p)}\Delta z} - \frac{(\rho w)_{wt} - (\rho w)_{wb}}{A_w^{(p)}\Delta z} .$$

Characteristically, the right-hand side of the obtained equality and, consequently, the right-hand side of the modified discrete equation (7) for the longitudinal velocity involve, among other things, the terms with a negative sign that contain the desired value of u_P. The fact that in such a case the role of constructing discrete analogs is violated [2] endows the computational equation with the property of conditional stability that does not guarantee the convergence of interactions in a general case. To obviate this serious shortcoming, it is necessary to transpose all the terms containing the sought velocity u_P into the left-hand side of the computational equation. The difference scheme will then become entirely implicit and will acquire the property of unconditional stability.

The same result can be obtained by a simpler method. It is necessary to add the terms containing the sought velocity u_P to both sides of the computational equation in order to neutralize the same terms entering with a negative sign into their right-hand side. Having made the needed mathematical transformations in Eqs. (7)–(9) for all three components of mass velocity, we obtain the discrete equations of the simplified algorithm:

$$a_P^{(u)} u_P = a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_B u_B + a_T u_T - \frac{p_e - p_w}{\Delta x} - \frac{D'_e/A_e^{(p)} - D'_w/A_w^{(p)}}{\Delta x} + \left(\frac{1}{A_w^{(p)}} + \frac{1}{A_e^{(p)}}\right)\frac{\rho_P u_P}{\Delta x^2}, \quad (10)$$

$$a_P^{(v)} v_P = a_W v_W + a_E v_E + a_S v_S + a_N v_N + a_B v_B + a_T v_T - \frac{p_n - p_s}{\Delta y} - \frac{D'_n/A_n^{(p)} - D'_s/A_s^{(p)}}{\Delta y} + \left(\frac{1}{A_s^{(p)}} + \frac{1}{A_n^{(p)}}\right)\frac{\rho_P v_P}{\Delta y^2}, \quad (11)$$

$$
\begin{aligned}
a_P^{(w)} w_P &= a_W w_W + a_E w_E + a_S w_S + a_N w_N + a_B w_B + a_T w_T \\
&- \frac{p_t - p_b}{\Delta z} - \frac{D_t'/A_t^{(p)} - D_b'/A_b^{(p)}}{\Delta z} + \left(\frac{1}{A_b^{(p)}} + \frac{1}{A_t^{(p)}} \right) \frac{\rho_P w_P}{\Delta z^2} .
\end{aligned} \tag{12}
$$

The coefficients a_P on the left-hand side of the computational equations are defined by the following formulas:

$$a_P^{(u)} = a_W + a_E + a_S + a_N + a_B + a_T + \rho_p \left(1/A_w^{(p)} + 1/A_e^{(p)}\right) \Big/ \Delta x^2 ,$$

$$a_P^{(v)} = a_W + a_E + a_S + a_N + a_B + a_T + \rho_p \left(1/A_s^{(p)} + 1/A_n^{(p)}\right) \Big/ \Delta y^2 ,$$

$$a_P^{(w)} = a_W + a_E + a_S + a_N + a_B + a_T + \rho_p \left(1/A_b^{(p)} + 1/A_t^{(p)}\right) \Big/ \Delta z^2 .$$

The appearance of mass sources in the discrete equations (10)–(12) for the velocity components has a definite physical sense: the difference between the mass sources favors the computational "passage" of liquid into the control volumes where the mass source is smaller. The convergence of iterations is ensured by the presence of a group of formally identical terms on both sides of the discrete equations (10)–(12), with these terms producing the under-relaxation of the calculated velocity component. This relaxation can be considered as locally definite, since at each computational node of the grid its value corresponds to the local condition of sustaining the convergence of iterations. As a result, the sum of absolute values of mass sources in the computational domain decreases in the process of iterations, whereas the velocity distribution better satisfies the differential continuity equation (1).

Thus, in the simplified algorithm intended for mathematical simulation of the three-dimensional motion of an incompressible fluid, the discrete equations (10)–(12), which determine the mass velocity vector, and one equation (4) for the pressure, are solved numerically. The discrete equation (5) for the correction to pressure is excluded from the mathematical model.

Generalized Form of Simplified Algorithm. It is worthwhile to introduce the locally determined relaxation into the equations for the velocity components also in those cases where the correction to pressure is calculated from discrete equation (5) under more rigorous boundary conditions. For example, it is possible to assume that the convergence of iterations in calculation of the velocity field will improve if the correction to pressure on the wall surface is equated to zero and simultaneously linked with the correction to pressure at the node closest to the grid. In particular, to establish connection between the pressure corrections p_s' and p_P' in the control volume set out in Fig. 2, it is necessary to find the coefficient A_S in Eq. (5) by the well-known formula [2]

$$A_S = \rho_s / (a_s^{(v)} \Delta y^2) ,$$

having preliminarily determined the coefficient $a_s^{(v)}$ in the displaced control volume for the transverse velocity v_s.

In this case, only one coefficient a_n on the faces of the displaced control volume differs from zero, which makes it possible to adopt the following estimation of the values of computational coefficients:

$$a_s^{(v)} = a_n \approx \rho_P \nu_P / \Delta y^2 , \quad A_S \approx 1/\nu_P ,$$

where ν_P is the sum of the kinetic coefficient of viscosity and its turbulent analog at the near-wall node of the grid.

Now, the computational discrete equations for the components of mass velocity are reduced to the following generalized form:

$$
\begin{aligned}
a_P^{(u)} u_P &= a_W u_W + a_E u_E + a_S u_S + a_N u_N + a_B u_B + a_T u_T \\
&- \frac{p_e - p_w + p_e' - p_w'}{\Delta x} + \left(\frac{1}{A_w^{(p)}} + \frac{1}{A_e^{(p)}} \right) \frac{\rho_P u_P}{\Delta x^2} ,
\end{aligned}
$$

$$a_P^{(v)} v_P = a_W v_W + a_E v_E + a_S v_S + a_N v_N + a_B v_B + a_T v_T$$

$$- \frac{p_n - p_s + p_n' - p_s'}{\Delta y} + \left(\frac{1}{A_s^{(p)}} + \frac{1}{A_n^{(p)}} \right) \frac{\rho_P v_P}{\Delta y^2} ,$$

$$a_P^{(w)} w_P = a_W w_W + a_E w_E + a_S w_S + a_N w_N + a_B w_B + a_T w_T$$

$$- \frac{p_t - p_b + p_t' - p_b'}{\Delta z} + \left(\frac{1}{A_b^{(p)}} + \frac{1}{A_t^{(p)}} \right) \frac{\rho_P w_P}{\Delta z^2} .$$

The coefficients a_P on the left-hand sides of these equations are determined in the same way as in Eqs. (10)–(12) of the simplified algorithm. In calculating the correction to pressure from discrete equation (5), it is not required now either to apply relaxation or organize the internal cycle of iterations.

Approval of the Simplified Computational Algorithm. The simplified algorithm of the numerical solution of the equations of incompressible fluid motion was presented for the first time at the conference in 1995 [5]. In subsequent years, it was improved and checked comprehensively. The application, over many years, of the simplified algorithm for computer simulation of subsonic motion of high-temperature gases has confirmed its reliable convergence in numerical solution of various engineering problems [6–9].

The experience of computer simulation has shown that the order and method of numerical solution of transformed discrete equations for pressure and velocity components does not exert a substantial effect on the calculated results. Nevertheless it is not recommended to apply any relaxation in solving a discrete equation for pressure, since for speeding up the convergence of iterations the pressure must correspond maximally to the velocity vector field.

CONCLUSIONS

1. For calculating the three-dimensional velocity field on a staggered grid a simplified algorithm has been justified that uses calculated mass sources for correcting the pressure field and that ensures unconditional convergence of iterations without solving the discrete equation for pressure correction.
2. The equations of the simplified algorithm are brought into a generalized form allowing one to introduce into them a pressure correction calculated from discrete equations at more rigorous boundary conditions.
3. A long-standing experience of applying the simplified algorithm has confirmed the reliable convergence of iterations in numerical simulation of subsonic motion of high-temperature gases in the approximation of incompressible fluid.

NOTATION

a, A, coefficients in equations for velocity and pressure components, respectively; D, free term in the discrete equation for pressure, kg/(m^3·s); D', calculated mass source taken with a reverse sign, kg/(m^3·s); p, static pressure, Pa; p', correction to pressure, Pa; u, v, w, velocity vector components, m/s; x, y, z, Cartesian coordinates, m; ν, kinematic coefficient of viscosity, m^2/s; ρ, density, kg/m^3. Indices: w, e, s, n, b, t, points on the faces of control volume; P, W, E, S, N, B, T, nodes of the grid; i, j = 1, 2, 3.

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