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Approximating a solution set of nonlinear inequalities

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Abstract In this paper we propose a method for solving systems of nonlinear inequalities with predefined accuracy based on nonuniform covering concept formerly adopted for global optimization. The method generates inner and outer approximations of the solution set. We describe the general concept and three ways of numerical implementation of the method. The first one is applicable only in a few cases when a minimum and a maximum of the constraints convolution function can be found analytically. The second implementation uses a global optimization method to find extrema of the constraints convolution function numerically. The third one is based on extrema approximation with Lipschitz under- and overestimations. We obtain theoretical bounds on the complexity and the accuracy of the generated approximations as well as compare proposed approaches theoretically and experimentally.

Keywords Systems of non-linear inequalities · Global optimization · Approximation · Robot's working area

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1 Introduction

Let nonempty set $X \subset \mathbb{R}^n$ be a solution of the following system of nonlinear inequalities

$$\begin{cases} g_j(x) \leq 0, \ j & \in \overline{1, m}, \\ a_i \leq x_i \leq b_i, & i \in \overline{1, n}, \end{cases} \tag{1}$$

where functions $g_j(x)$ are continuous.

Such systems arise in many applications and have been studied extensively [10,12,28]. For instance, in [10] the authors transform Problem (1) into an inequality constrained optimization problem and solve it by using the filter method [7]. The authors of [12] assume that $G(x) = (g_1(x), \ldots, g_m(x))^T$ has Lipschitz continuous Jacobian, transform Problem (1) into an equality and solve it by using a smoothing-type algorithm. A similar approach is used in [28].

Methods proposed in [10,12,28] are aimed at finding one or several points satisfying the system of inequalities. However, there are many applications where it is necessary to get a whole set of solutions. Heuristic methods for generating a tetrahedral 3-D mesh approximating the solution set of non-linear inequalities are suggested in [8]. Papers [15,16,19] propose and study polyhedral approximations of convex sets. Approximations of sets defined as images of continuous mapping of compact sets are described in [4].

Interval analysis techniques [11,21] can also be applied to approximating the solution set of non-linear equalities and inequalities. See [13,14,22] for examples. The approach most close to ours is described in [13,14]. Authors consider a set-inversion problem stated as follows. Given a continuous mapping $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a set $Y \subseteq \mathbb{R}^m$ approximate the set $f^{-1}(Y)$. Authors developed a general SIVIA (Set Inversion via Interval Analysis) algorithm that constructs an inner and an outer approximations of $f^{-1}(Y)$ set. The approximations consist of non-intersecting boxes. It was shown that under certain assumptions at the limit the approximations produced by SIVIA converge to the target set. Obviously the problem (1) can be treated as a particular case of the set inversion problem (see [13] for details).

The main difference between SIVIA and our method is that we do not limit our approach to interval analysis and study in deep two special cases where the bounds are computed using a global optimization techniques (Sect. 4) or by Lipschitzian estimations (Sect. 5). For these two cases we derive formulas characterizing the precision of constructed approximations. Such formulas are hard to obtain for interval bounds since the accuracy of such bounds is difficult to estimate. It should be noted that the Lipschitzian techniques has two advantages over the interval analysis approach. The first advantage is that Lipschitzian constants can be efficiently approximated [17,26] and thus can be used for "black-box" problems with no analytical representation of the constraints. Such problems obviously can't be handled by the interval methods. The second advantage is that Lipschitzian techniques can be used for sets with more complex shape than boxes e.g. simplices or other convex polytopes where the use of interval bounds is problematic [24]. The latter approach seems to be quite prospective since polyhedral approximations are more accurate than box-based ones.

The objective of this paper is to describe the solution set for Problem (1) with guaranteed accuracy. It worths noting that the solution set can be non-convex and disjointed. The problem under consideration has a lot in common with the deterministic global optimization [1,3,5,17,18] where the goal is to find a solution and prove its optimality to a certain degree of accuracy. Below we show how global optimization techniques elaborated in [3,5] can be tailored to approximate the solution set of system of inequalities with the given precision.

This work continues the research started in [2,6]. In this paper we present a rigorous theory, estimate the complexity and prove new results regarding the accuracy of the approximations. We also perform a deep experimental study of the proposed approach on a set of new examples including workspace assessment of the real-life planar parallel robot.

In Sect. 2 we formulate the general method to solve System (1) based on the nonuniform covering techniques [3,5], and consider its main properties. In Sect. 3 we propose an algorithm to solve the system, discuss its properties and complexity. In Sects. 4 and 5 two different ways of estimating the constraints in (1) are considered. The first method is based on a global optimization technique and the second one relies on Lipschitzian over- and underestimations. Numerical results are reported in Sect. 6, where we consider a model example and a practical application of the proposed approach. Section 7 concludes the paper.

2 Outline of the proposed approach

First notice that the system (1) is equivalent to the following one:

$$\begin{cases} \phi(x) \leq 0, \\ x \in P, \end{cases} \tag{2}$$

where $\phi(x) = \max_{j=1,\ldots,n} g_j(x)$ is a convolution of constraints and $P = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i \in \overline{1,n}\}$ is an n-dimensional bounding box. The goal is to find X with some predefined accuracy.

Our method for approximating the solution set of (2) exploits the idea of the non-uniform covering approach adopted for global optimization [3,5]. The set of boxes $\{P_i\}$, $i = \overline{1,k}$ is called a *coverage* if

$$P = \cup_{i \in \overline{1,k}} P_i, \tag{3}$$

and each box $P_i, i \in \overline{1,k}$ satisfies one of the following statements:

$$\max_{x \in P_i} \phi(x) < 0, \tag{4}$$

$$\min_{x \in P_i} \phi(x) > 0, \tag{5}$$

$$\text{neither (5) nor (4) holds and } d\,(P_i) \leq \delta, \tag{6}$$

where $d(P_i) = \sup\{\|x_1 - x_2\|, x_1, x_2 \in P_i\}$ is the diameter of the box and $\delta > 0$ is the accuracy of the approximation.

From the formulations above we can infer the following

Proposition 1 *A box $Q \subseteq P$ is a subset of X iff statement (4) holds for this box. A box $Q \subseteq P$ has no common points with X iff statement (5) holds for this box.*

Let $I \subseteq \overline{1,k}$ be the index set of all boxes from the coverage satisfying (4), $Q_I = \cup_{i \in I} P_i$, $E \subseteq \overline{1,k}$ be the index set of all boxes satisfying (5), $Q_E = \cup_{i \in E} P_i$ and $B \subseteq \overline{1,k}$ be the index set of all boxes satisfying (6), $Q_B = \cup_{i \in M} P_i$.

Proposition 2 *It holds that*

$$Q_I \subseteq X \subseteq Q_I \cup Q_B \tag{7}$$

Proof The first part of the proposition: $Q_I \subseteq X$ is true due to statement (4) for all boxes in Q_I. The second part follows from two obvious observations: $Q_I \cup Q_B \cup Q_E = P$ and (according to Proposition 1) $Q_E \cap X = \emptyset$. □

Definition 1 The set of points x of $\mathbb{R}^n$ such that every neighborhood of x contains at least one point of X and at least one point not of X is called a *boundary of a set* X and is denoted as ∂X.

Definition 2 A set X defined by (1) is *regular*, if its boundary is the set of all x such that $\phi(x) = 0$.

This definition implies that a regular set is completely defined by the inequality $\phi(x) \leq 0$ and the box P encloses this set.

Proposition 3 *For a boundary ∂X of a regular set X it holds that $\partial X \subseteq Q_B$.*

Proof Suppose it is not true, i.e. there is a point $x \in \partial X$, which is not in Q_B, therefore $x \in Q_I$ or $x \in Q_E$. Considering (4) and (5), we can conclude that $\phi(x) \neq 0$. On the other hand, for any point $x \in \partial X$ it holds that $\phi(x) = 0$, which contradicts with our previous conclusion, and as a consequence with the initial guess. □

Proposition 4 *If X is a regular set then every box satisfying equation (6) contains at least one point of the boundary ∂X.*

Proof Let Q be a box satisfying (6). Therefore properties (4), (5) do not hold for Q. Thus there exist two points $x, y \in Q$ such that $\phi(x) \geq 0$ and $\phi(y) \leq 0$. Since the function ϕ is the continuous one the function $\psi(t) = \phi(x + t(y - x)), t \in \mathbb{R}$ is also continuous. Obviously $\psi(0) = \phi(x) \geq 0$, $\psi(1) = \phi(y) \leq 0$. According to the mean value theorem there is a point $\hat{t} \in [0, 1]$ such that $\psi(\hat{t}) = 0$. Thus $\phi(\hat{z}) = 0$, where $\hat{z} = x + \hat{t}(y - x)$. Taking into account that Q is convex and $\hat{z}$ lies on a segment $[x, y]$ we conclude that $\hat{z} \in Q$. Since the set X is regular, $\hat{z}$ is a point on its boundary. □

Definition 3 A δ-*neighborhood* of a set $A \subseteq \mathbb{R}^n$ is a set $B(A, \delta) = \cup_{a \in A} B(a, \delta)$, where $B(a, \delta) = \{x \in \mathbb{R}^n : \|x - a\| \leq \delta\}$ is a δ-*neighborhood* of $a \in \mathbb{R}^n$.

Definition 4 Let $X^{-\delta} = X \backslash B(\partial X, \delta)$ be the subset of X obtained by excluding the boundary of X with its δ-neighborhood and $X^{+\delta} = X \cup B(\partial X, \delta)$ be the union of X and its δ-neighborhood.

Corollary 1 *For every regular set X it holds that*

$$Q_B \subseteq B(\partial X, \delta). \tag{8}$$

Proof Let $P_i \subseteq Q_B, i \in B$ be a box from coverage (3). The Proposition 4 states that there is a point $x \in P_i \cap \partial X$. Since $d(P_i) \leq \delta$ we obtain $P_i \subseteq B(\partial X, \delta)$. □

Theorem 1 *For every regular set X it holds that*

$$X^{-\delta} \subseteq Q_I \subseteq X \subseteq Q_I \cup Q_B \subseteq X^{+\delta}. \tag{9}$$

Proof Applying Corollary 1, we can conclude that the following two parts of Theorem 1 are true: $X^{-\delta} \subseteq Q_I$ and $Q_I \cup Q_B \subseteq X^{+\delta}$. This fact and Proposition 2 yield the desired result. □

From Theorem 1 it follows that if we have constructed a coverage $\{P_i\}$ satisfying properties (4)–(6) then sets Q_I and $Q_I \cup Q_B$ serve as internal and external δ-approximations of the set X respectively. In Sect. 3 we describe the algorithm to construct a coverage.

3 An algorithm for approximating the solution set of a system of nonlinear inequalities

In this section we propose an algorithm for numerical approximation of a solution set for System (1) based on the approach described in Sect. 2. The idea is simple: we partition the initial box into smaller boxes until at least one of the properties (4)–(6) fulfills. The partitioning is done by the longest side of a box.

The *Covering Algorithm* depicted (see Algorithm 1) works as follows. On the initial step two lists of boxes L_{main} and L_{temp} are initialized. The former is initialized with P and is used on every iteration of the the algorithm as the main storage for boxes to be analyzed. The latter is a temporal storage for boxes obtained by splitting ones from L_{main}. The diameter of the initial box P is saved to δ_c. Three empty lists L_I, L_E and L_B for storing boxes contributing to the sets Q_I, Q_E and Q_B respectively are created.

On each iteration of the *while* loop we traverse and analyze all boxes of the main list L_{main} in the *for* loop. On i-th iteration of the *for* loop a box P_i is processed. If inequality (5) holds for P_i, the box is added to the list of external boxes: L_E. If inequality (4) holds, the box is added to the list of internal ones: L_I. If neither (4) nor (5) hold, the box is partitioned into two equal boxes P^l_q and P^r_q by bisecting its longest edge. The diameter of the box P^l_q (same as the diameter of P^r_q) is calculated and δ_c value is updated. The algorithm stops when δ_c is less then δ. After termination the list L_{main} is equal to the list L_B by construction.

The partitioning of boxes is done by bisecting its longest edge thereby ensuring a sufficient reduction of its diameter.

Proposition 5 *Let boxes B' and B'' are obtained from the box B by bisecting its longest edge. Then $d(B') = d(B'') \leq \alpha d(B)$, where $\alpha = \sqrt{1 - \frac{3}{4n}}$.*

Proof The diameter of a box B is computed as follows:

$$d(B) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

Then the diameter of the box B' obtained by bisecting the longest edge q of the box B is

$$d(B') = \sqrt{(b_1 - a_1)^2 + \cdots + \frac{(b_q - a_q)^2}{4} + \cdots + (b_n - a_n)^2}.$$

Then, $d(B')^2 = d(B)^2 - \frac{3}{4}(b_q - a_q)^2$.

Since $|b_q - a_q| \geq |b_i - a_i|$ for any $i \neq q$, we can write that $d(B)^2 \leq n(b_q - a_q)^2$, which gives us the following inequality for $d(B')$:

$$d(B') = \sqrt{d(B)^2 - \frac{3}{4}(b_q - a_q)^2} \leq \sqrt{d(B)^2 - \frac{3}{4}\frac{d(B)^2}{n}} = d(B)\sqrt{1 - \frac{3}{4n}},$$

which proves the proposition. □

Proposition 5 can be used to estimate the maximal number of steps of the algorithm. Let's define the *complexity* of the Covering Algorithm as the total number of iterations.

Theorem 2 *Let d be a diameter of the initial box and δ be and accuracy of the approximation. Then, the complexity of the Covering Algorithm is less then*

$$\left(\frac{\delta}{d}\right)^{2/\log_2(1-\frac{3}{4n})}, \tag{10}$$

Algorithm 1: The Covering Algorithm. The pseudocode for the algorithm of getting a solution set for the problem (1). It returns the list of boxes: L_I, L_E, and L_B that storing boxes from Q_I, Q_E and Q_B respectively.

```
Input: P, δ
Output: L_I, L_E, L_B
L_main ← P;
L_temp := ∅;
δ_c := d(P);
L_I := ∅; L_E := ∅; L_B := ∅;
while δ_c ≥ δ do
    foreach P_i ∈ L_main do
        if min_{x∈P_i} φ(x) > 0 then
            L_E ← P_i;
            continue with another node from L_main;
        end
        if max_{x∈P_q} φ(x) < 0 then
            L_I ← P_i;
            continue with another node from L_main;
        end
        Split P_q into two: P_i^l ? P_i^r
        L_temp ← P_i^l;
        L_temp ← P_i^r;
        δ_c := d(P_i^l);
    end
    L_main := L_temp;
    L_temp := ∅;
end
L_B := L_main
return L_I, L_E, L_B
```

Proof Applying Proposition 5, we obtain the following inequality for the k-th iteration of the *while* loop in Algorithm 1:

$$\alpha^{k-1} d \geq \delta, \tag{11}$$

where $\alpha = \sqrt{1 - \frac{3}{4n}}$

It can be rewritten as follows:

$$k \leq \log_\alpha \left(\frac{\delta}{d}\right) + 1 = \log_2 \left(\frac{\delta}{d}\right) / \log_2 \alpha + 1 \tag{12}$$

Starting from 1 the number of boxes in the list L_{main} is (in the worst case) doubled on each iteration of the *while* loop. Since the number of iterations of the *for* loop is equal to the size of the list L_{main}, the upper bound of the algorithm complexity S can be evaluated as follows:

$$S \leq 2^{k-1} \leq \left(\frac{\delta}{d}\right)^{1/\log_2 \alpha} = \left(\frac{\delta}{d}\right)^{2/\log_2 (1-\frac{3}{4n})}, \tag{13}$$

which gives us a desired result. □

After correct termination of the Covering Algorithm, we get two lists of boxes L_I, L_B comprising boxes of the sets Q_I and Q_B respectively. Therefore, we obtain inner and outer approximations of the set X in the sense of the Theorem 1.

Although the approach described above can be used in practice, it works only in simple cases when we can compute exact maximum and minimum of the function $\phi(x)$ for a box. If it is problematic, we have to use more elaborate approaches that we develop in the next sections.

4 The solution set approximation using global optimization techniques

In this section we consider the case when extrema of the function $\phi(x)$ are found using global optimization techniques. To provide correctness we obviously need deterministic methods that guarantee the precision of the obtained solutions. As a rule such algorithms yield a solution x^{ε} of the problem $\max_{x\in P_i}\phi(x)(\min_{x\in P_i}\phi(x))$ with a predefined accuracy ε:

$$|\phi(x^{\varepsilon})-\phi(x^{*})|\le\varepsilon,\ \ \varepsilon\ge 0, \tag{14}$$

where $x^{\varepsilon}\in P_i$ denotes a point obtained using the global optimization techniques, and x^{*} denotes a point where a global extremum of the function ϕ is achieved.

Based on the Eq. (14), we can rewrite Eqs. (4) and (5) as follows:

$$\max_{x\in P_i}{}^{\varepsilon}\phi(x) < -\varepsilon, \tag{15}$$

$$\min_{x\in P_i}{}^{\varepsilon}\phi(x) > \varepsilon, \tag{16}$$

$$\text{neither (15) nor (16) holds and } d\,(P_i)\le\delta, \tag{17}$$

where $\max_{x\in P_i}{}^{\varepsilon}\phi(x)$ and $\min_{x\in P_i}{}^{\varepsilon}\phi(x)$ are extrema of the function $\phi(x)$ found by a global optimization method that satisfies (14) for a box P_i.

The coverage (3) as well as the sets Q_I, Q_E, and Q_B are redefined according to (15), (16), (17).

Propositions 2 and 3 hold for this case, while Proposition 4 and Theorem 1 do not. We can modify them as follows.

Definition 5 The ε-boundary of X is $\partial_{\varepsilon}X=\{x\in\mathbb{R}^n : -\varepsilon\le\phi(x)\le\varepsilon\}$.

Proposition 6 *If X is a regular set then every box satisfying (17) contains at least one point from $\partial_{\varepsilon}X$.*

Proof Let P_i be a box satisfying (17). Therefore (15) doesn't hold i.e. $\max_{x\in P_i}{}^{\varepsilon}\phi(x)\ge-\varepsilon$. Thus there exists $p\in P_i$ such that $\phi(p)\ge-\varepsilon$. If $\phi(p)\le\varepsilon$ then $p\in P_i\cap\partial_{\varepsilon}X$ and the proposition is proved. Inequality (16) is also false. Thus $\min^{\varepsilon}_{x\in P_i}\phi(x)\le\varepsilon$ and there exists $q\in P_i$ such that $\phi(q)\le\varepsilon$. If $\phi(q)\ge-\varepsilon$ then $\phi(q)\in P_i\cap\partial_{\varepsilon}X$ and the proposition is proved. The remaining case $\phi(p)>\varepsilon>0>-\varepsilon>\phi(q)$ is considered in a way similar to the proof of Proposition 4. In that case we construct $z\in P_i$ such that $\phi(z)=0$. Obviously $z\in P_i\cap\partial_{\varepsilon}X$. □

Since the diameter of boxes in Q_B is less than δ, the Proposition 6 yields the following

Corollary 2 *Q_B is a subset of $B(\partial_{\varepsilon}X,\delta)$.*

Define sets $X^{-\delta}_{\varepsilon}$ and $X^{+\delta}_{\varepsilon}$ as follows: $X^{-\delta}_{\varepsilon}=X\setminus B(\partial_{\varepsilon}X,\delta)$, $X^{+\delta}_{\varepsilon}=X\cup B(\partial_{\varepsilon}X,\delta)$. The following theorem is the direct consequence of the Corollary 2.

Theorem 3 *For a regular set X it holds that*

$$X_\varepsilon^{-\delta} \subseteq Q_I \subseteq X \subseteq Q_I \cup Q_B \subseteq X_\varepsilon^{+\delta}. \tag{18}$$

The Covering Algorithm can be rewritten to be used with global optimization techniques with a predefined accuracy ε by replacing inequalities (4), (5) with (15), (16) respectively. The other parts of the algorithm remain the same.

5 The solution set approximation using Lipschitz minorants and majorants

The drawback of the approach described in the previous section is that it relies on computationally intensive global optimization procedures. To cope with this issue, here we are going to approximate the optima of $\phi(x)$ by using Lipschitzian over- and underestimations.

Let functions $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in \overline{1,m}$ in (1) satisfy the Lipschitz continuity property:

$$|g_j(x) - g_j(y)| \le L_j \|x - y\|, \quad j \in \overline{1,m}, \tag{19}$$

where $P = [a, b] = \{x : a_i \le x_i \le b_i, i \in \overline{1,n}\}$ is a box in $\mathbb{R}^n$, the diameter of which is $d(P) = \|b - a\|$. Obviously function $\phi(x)$ is Lipschitzian with the constant $L = \max_{j\in\overline{1,m}} L_j$.

The Lipschitz continuity of the function ϕ can be used to get underestimation $\mu_i(x)$ and overestimation $M_i(x)$ for a box $P_i = \left[a^{(i)}, b^{(i)}\right]$:

$$\mu_i(x) = \phi\left(c^{(i)}\right) - L\left\|x - c^{(i)}\right\|, M_i(x) = \phi\left(c^{(i)}\right) + L\left\|x - c^{(i)}\right\|, \tag{20}$$

where $c^{(i)} = \frac{1}{2}\left(a^{(i)} + b^{(i)}\right)$ is the center of the box P_i.

Applying the property of this approximations: $\mu_i(x) \le \phi(x) \le M_i(x)$, $x \in P_i$, we can modify (4) and (5) as follows.

For any box from Coverage (3) one of the following properties hold:

$$\max_{x\in P_i} M_i(x) < 0, \tag{21}$$

$$\min_{x\in P_i} \mu_i(x) > 0, \tag{22}$$

$$\text{neither (21) or (22) holds and } d(P_i) \le \delta. \tag{23}$$

In this case, extrema of the minorant and the majorant can be found analytically:

$$\begin{aligned} \max_{x\in P_i} M_i(x) &= \phi\left(c^{(i)}\right) + L\, d(P_i)/2, \\ \min_{x\in P_i} \mu_i(x) &= \phi\left(c^{(i)}\right) - L\, d(P_i)/2, \end{aligned} \tag{24}$$

This fact enables checking conditions (21), (22) without using global optimization techniques.

The coverage (3) as well as the sets Q_I, Q_E, and Q_B are redefined according to (21), (22), (23). While Propositions 2 and 3 remains intact for the case under consideration, Theorem 1 must be revised.

Lemma 1 *If* (23) *holds for a box* P_i, *then its center* $c^{(i)} \in \partial_\varepsilon X$ *where* $\varepsilon = \frac{L\delta}{2}$.

Proof Since (23) holds for P_i, $\phi\left(c^{(i)}\right) - \varepsilon \leq 0 \leq \phi\left(c^{(i)}\right) + \varepsilon$. Therefore, $-\varepsilon \leq \phi\left(c^{(i)}\right) \leq \varepsilon$ which proves the lemma. □

Corollary 3 *A set Q_B is the subset of $B(\partial_\varepsilon X, \delta)$ where $\varepsilon = \frac{L\delta}{2}$.*

Based on this additional information, we can formulate the following analog of Theorem 1.

Theorem 4 *It holds that*

$$X_\varepsilon^{-\delta} \subseteq Q_I \subseteq X \subseteq Q_I \cup Q_B \subseteq X_\varepsilon^{+\delta}, \tag{25}$$

where $\varepsilon = \delta L/2$.

Proof Applying the Corollary 3, we can conclude that the following statements are true: $X_\varepsilon^{-\delta} \subseteq Q_I$ and $Q_I \cup Q_B \subseteq X_\varepsilon^{+\delta}$. Considering this fact and Proposition 2 we can get the desired result. □

The Covering Algorithm can be rewritten to be used with the Lipschitzian under- and overestimations by applying Eq. (24) to solve analytically the optimization problem in (4) and (5). The other parts of the algorithm remain the same.

6 Numerical experiments

In this section we apply the approach proposed in Sect. 2 to different systems of inequalities. We implemented and compared two algorithms based on global optimization and extrema approximation techniques discussed in Sect. 4 and Sect. 5 respectively. We used Lipschitzian optimization techniques proposed in [3,5] as a global optimizer needed for the algorithm described in Sect. 4.

All experiments were run on a laptop with a 2.2 GHz Intel Core i7 processor and 16 GB of RAM, running MacOS X (10.10.5). We made the developed software written in Python programming language publicly available from the following GitHub repository https://github.com/andreiturkin/JGO2018.SolSetApp.git.

6.1 Example 1

Consider the problem (1) with the following constraint functions [10,12,28]:

$$\begin{cases} g_1(x_1, x_2) = x_1^2 + x_2^2 - 1, \\ g_2(x_1, x_2) = (0.999)^2 - x_1^2 - x_2^2, \\ -1.5 \leq x_1 \leq 1.5, -1.5 \leq x_2 \leq 1.5. \end{cases} \tag{26}$$

Elaborate approaches to estimating Lipschitz constant's value are considered in [26,27]. Though such approaches proved their high efficiency for practical (especially black-box) problems they do not guarantee the validity of the estimated Lipschitz constant. To provide the precision guarantee of the constructed approximations of the solution set X we need a valid Lipschitz constant. Here we can take an advantage of the available analytic representation of constraint functions $g_i(x), i = 1, \ldots, m$. For a differentiable function $f(x)$ any number above $\max_{x \in P} \|\nabla f(x)\|$ is a valid Lipschitz constant for $f(x)$ [23]. Such upper bound can be obtained using interval analysis [11] or other approximation techniques.

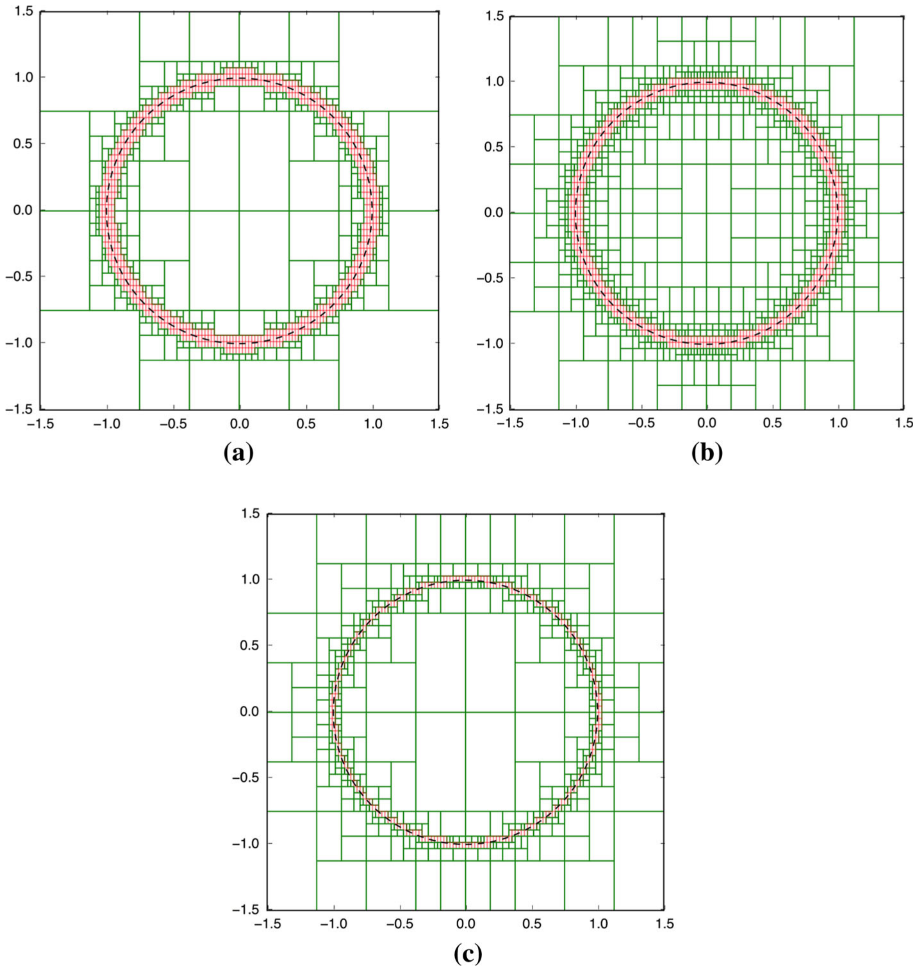

Fig. 1 The approximations are calculated for $\delta = 0.06$ and $\varepsilon = \delta L/2$ by using global optimization (**a**), the extrema approximation (24) with the L value determined for the initial rectangle (**b**), and the extrema approximation with the L value calculated for every box of the coverage (3) (**c**). **a** The approximation is calculated by using global optimization. **b** The approximation is calculated with the "global" L value determined for the initial box. **c** The approximation is calculated by using locally recalculated values of Lipschitz constants

In this example, the Lipschitz constant can be found analytically as a maximum value of the gradient norm by using the following equation:

$$L = 2\big(\max(|a_1|, |b_1|)^2 + \max(|a_2|, |b_2|)^2\big)^{1/2}. \tag{27}$$

Figure 1 shows the results obtained for $\delta = 0.06$. The blue color marks the internal approximation of the solution set Q_I. Rectangles with red and green boundaries belong to the sets Q_B and Q_E respectively. Figure 1a depicts the approximation obtained by applying the global optimization technique proposed in [3,5] with $\varepsilon = \delta L/2$. The approximation obtained by applying under- and overestimations (20) is presented in Fig. 1b. In both cases we used global bounds for Lipschitz constants L calculated according to (27) for the initial box. The results for locally recalculated Lipschitz constants are presented at Fig. 1c. In the

Table 1 The relationship between the δ value and the run-time of the Algorithm 1 for different optimization techniques used to find a minimum value for the function $\phi(x_1, x_2)$

δ	Global optimization		Global L^*		LocalL^{**}	
	Number of iterations	Time (s)	Number of iterations	Time (s)	Number of iterations	Time (s)
0.08	1159	0.2675	1983	0.0716	1015	0.0370
0.06	1671	0.3975	2831	0.0974	1391	0.0600
0.035	2215	0.6418	4087	0.1491	1959	0.0790
0.03	3463	0.8398	5735	0.2101	2703	0.1132
0.018	4287	1.2530	8343	0.2867	3879	0.1457
0.015	6803	1.7493	11,655	0.4038	5471	0.2311
0.01	9383	2.6085	16,911	0.5856	7975	0.3330
0.001	152,959	39.0388	231,447	8.6138	132,063	4.9824

* The Lipschitz constant is calculated only for the initial box P by using (27)
** The Lipschitz constant is calculated for every box P_i in (3)

latter case the Lipschitz constants were recomputed according to (27) for every processing box.

As it is shown in Fig. 1a, b, if a global optimization technique with $\varepsilon = \delta L/2$ is used to find a minimum for a box, we get a smaller number of rectangles in Q_E. On the other hand (see Table 1), the run-time of the Algorithm 1 with the global optimization technique above is longer than the one with Lipschitzian under- and overestimations. This fact is caused by time-consuming global optimization techniques invoked at every step of the method.

Figure 1 and Table 1 demonstrate that the use of locally calculated Lipschitz constants improves the quality of the approximation and reduces the running time approximately twice. Thus the best performance is obtained when a combination of Lipschitzian under- and over-estimations combined with the local estimation of Lipschitz constants is used. Thus We focus on this combination in the rest of the paper.

The solution set of the problem (26) is a very thin ring. To capture its internity we need to select a small $\delta = 10^{-4}$ or less. The Fig. 2 shows the result of computations. The left part of Fig. 2 is the whole solution set of (26), in Fig. 2b we plot a zoomed in portion of the solution set shown in blue.

6.2 Example 2

The following example shows an ability of the proposed approach to approximate the solution set when constraints have a more complex structure:

$$\begin{cases} g_1(x_1, x_2) = x_1 \sin(x_1) + 0.1x_1^2 + 1, \\ g_2(x_1, x_2) = \cos(x_2) + 0.1x_2^2, \\ \pi \le x_1 \le 2\pi, 1 \le x_2 \le \pi + 1. \end{cases} \tag{28}$$

We implemented two approaches for this example (Table 2). The first one relies on valid analytic upper bounds for the Lipschitz constant. The second uses the heuristically estimated Lipschitz constants. The valid Lipschitz constant can be found by using interval arithmetic from the following equation:

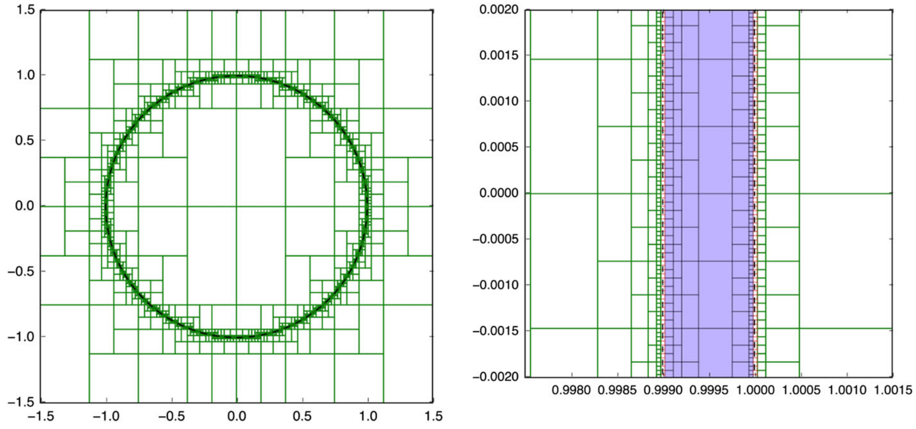

Fig. 2 The approximations are calculated for $\delta = 0.0001$ and $\varepsilon = \delta L/2$. **a** The whole ring of the solution set. **b** A zoomed in portion of the solution set near the point (0.9995, 0)

Table 2 The table presents the relationship between the δ value and the run-time of the Algorithm 1 for two approaches to Lipschitz constant calculation

δ	Analytically calculated L*		Numerically calculated L**	
	Number of iterations	Time	Number of iterations	Time (s)
0.25	557	3.9250	513	2.2362
0.1	1989	15.1308	1947	7.4649
0.05	3953	26.7782	3963	15.4814
0.03	5393	37.3575	5571	22.7692
0.018	7677	55.5041	8179	31.5296
0.015	10,723	72.7264	11,141	42.9232
0.01	15,407	104.1031	15,543	60.0644
0.005	30,657	206.6940	30,579	120.1384

* The Lipschitz constant is calculated for every box by using analytical representation and interval arithmetic
** The Lipschitz constant is calculated for every box by using second order accurate central differences to get numerically norm of the function gradient

$$L = \max_{(x_1,x_2)\in P_i} \Big(|\sin x_1 + x_1 \cos x_1 + 0.2x_1|, |-\sin x_2 + 0.2x_2| \Big). \tag{29}$$

The resulting approximation is depicted in Fig. 3a.

Figure 3b demonstrates the results obtained with the heuristically estimated Lipschitz constants. The goal of this experiment is to demonstrate the ability of the proposed approach to cope with “black-box” problems, e.g. when the analytic representation of the constraint functions is not available or too complex to handle.

We use a very basic approach to Lipschitz constant estimation based on the following obvious observation [25]:

$$L(P_i) \geq \max_{i=1,\ldots,N} \max_{j=i+1,\ldots,N} \frac{|f(x_i) - f(x_j)|}{\|x_i - x_j\|}, \tag{30}$$

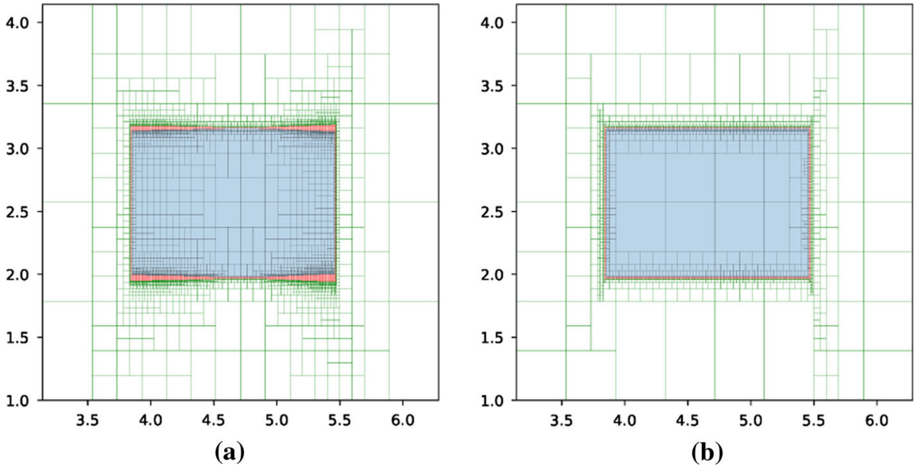

Fig. 3 The approximations are calculated for $\delta = 0.01$ and the L values that are calculated by using either (29) (**a**) or (31) (**b**)

where $L(P_i)$ is a valid Lipschitz constant for the box P_i and $x_1, \ldots, x_N \in P_i$ are sample points. Based on the inequality (30) we derive the following estimation of the Lipschitz constant:

$$L = \alpha \max_{i=1,\ldots,N} \max_{j=i+1,\ldots,N} \frac{|f(x_i) - f(x_j)|}{\|x_i - x_j\|}, \tag{31}$$

where α accounts the anticipated difference between left and right parts of the inequality (30). In this experiment we used $N = 10$ and $\alpha = 1.2$.

As we can see from comparison of Fig. 3a, b the heuristic estimations of Lipschitz constants give the approximation sufficiently close to one obtained with analytic estimations. This observation opens broad opportunities for applying the proposed method to "black-box" problems. Notice that more sophisticated techniques for estimating Lipschitz constants can be employed [24,27].

6.3 Example 3

In this example the proposed approach was applied to solve a practical problem from the field of robotics: the workspace assessment [20]. This problem consists in describing the set of all possible states of the robot's end-effector (i.e. working tool). This problem can formulated as a system of nonlinear inequalities, which has a general form (1).

Consider the planar robot described in [9]. This robot presented in Fig. 4 has linear actuators 1, 2, 3 that are used to change lengths of links ($\overline{A_1B_1}$, $\overline{A_2B_2}$, and $\overline{A_3B_3}$), correspondingly, thereby moving the triangular platform with its center Z in a desired direction.

The workspace assessment problem can be formulated as follows: it is necessary to find an approximation of the solution set $X \subset \mathbb{R}^2$ for system (1), where functions $g_j(\cdot)$, $j \in \overline{1,6}$ are defined as follows:

$$\begin{aligned} g_{2u-1}(x_1, x_2) &= (x_1 - p_u(\xi))^2 + (x_2 - q_u(\xi))^2 - (l_u^{max})^2, \\ g_{2u}(x_1, x_2) &= (l_u^{min})^2 - (x_1 - p_u(\xi))^2 - (x_2 - q_u(\xi))^2, \\ u &= 1, 2, 3 \end{aligned} \tag{32}$$

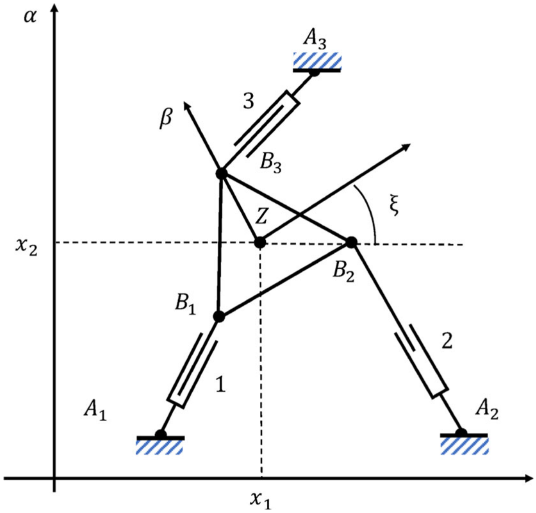

Fig. 4 The three degree of freedom planar parallel manipulator

Table 3 The table of parameters [9]

Parameter	u = 1	u = 2	u = 3
x_1^u	-15	15	0
x_2^u	$5\sqrt{3}$	$-5\sqrt{3}$	$10\sqrt{3}$
$\hat{x}_1^u$	-5	5	0
$\hat{x}_2^u$	$-5\sqrt{3}/3$	$-5\sqrt{3}/3$	$10\sqrt{3}/3$
l_u^{min}	12	12	12
l_u^{max}	27	27	27

where x_1, x_2 are the coordinates of the platform center, l_u^{min} and l_u^{max} denote the intervals for possible lengths of $\overline{A_u B_u}$: $\|\overline{A_u B_u}\| \in [l_u^{min}, l_u^{max}]$;

$$p_u(\xi) = x_1^u - \hat{x}_1^u \cos\xi + \hat{x}_2^u \sin\xi,$$
$$q_u(\xi) = x_2^u - \hat{x}_1^u \sin\xi - \hat{x}_2^u \cos\xi, \tag{33}$$

where x_1^u and x_2^u are coordinates of points A_u in the frame α, and $\hat{x}_1^u$ and $\hat{x}_2^u$ are coordinates of points B_u in the moving frame β. We restrict our consideration to the constant orientation workspaces, i.e. when the angle ξ is fixed at some value. Therefore, in this case, the values of $p_u(\xi), q_u(\xi)$ are also constants. The values of parameters for this example are summarized in Table 3. The bounding box was chosen sufficiently large to reliably capture the workspace: $-20 \le x_1 \le 20$, $-20 \le x_2 \le 20$.

Applying the same approach as in the previous example to calculate the Lipschitz constant for $\phi(x_1, x_2)$ and some rectangle P_i obtain:

$$L = 2 \max_u \Big(\max_{l \in \{a_1, b_1\}} (|l - p_u(\xi)|)^2 + \max_{l \in \{a_2, b_2\}} (|l - q_u(\xi)|)^2 \Big)^{1/2} \tag{34}$$

Figure 5 depicts four coverages with different values of δ that were obtained by using Lipschitzian minorants and majorants with the L value recalculated locally for every processed box. In Fig. 5 we show the coverage obtained for $\delta = 0.1$ and the following ξ values: $\xi_1 = 50$ (Fig. 5a); $\xi_2 = 80$ (Fig. 5b); $\xi_3 = 120$ (Fig. 5c); and $\xi = 140$ (Fig. 5d). The workspace is the

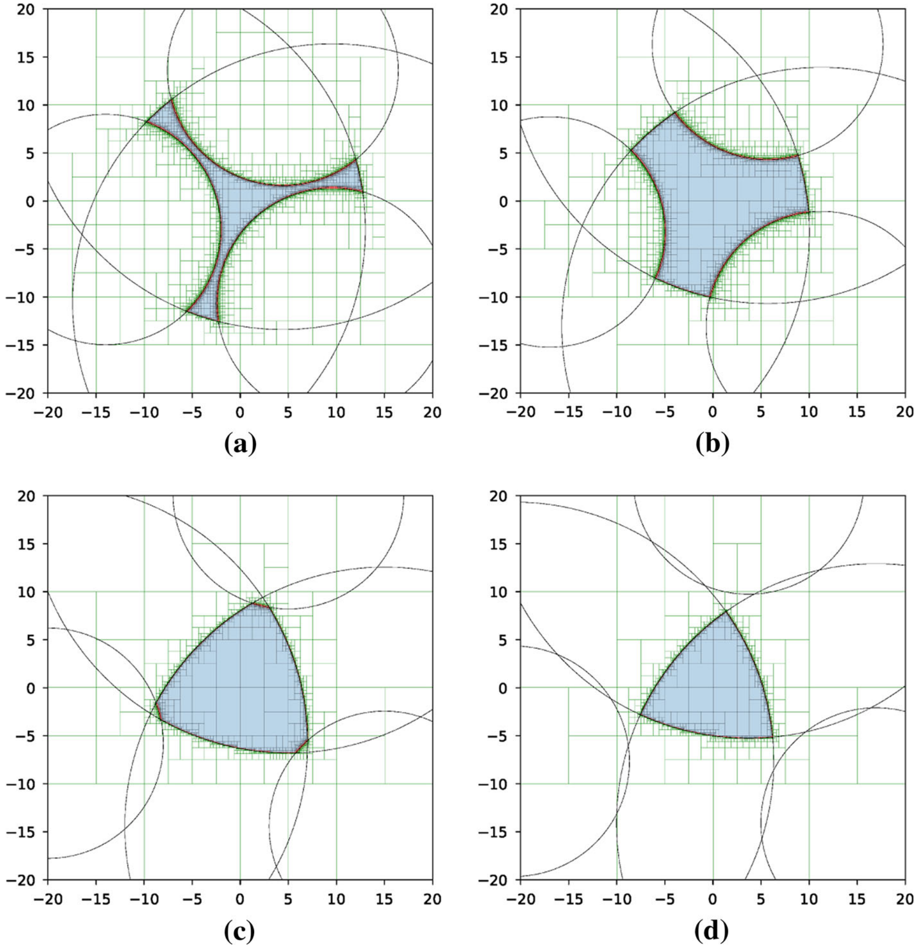

Fig. 5 The approximations are calculated for $\delta = 0.1$ and $\xi \in \{50, 80, 120, 140\}$. **a** $\xi_1 = 50$. **b** $\xi_2 = 80$. **c** $\xi_3 = 120$. **d** $\xi_4 = 140$

Table 4 The relationship between the δ value and the run-time for the Algorithm 1 applied to the problem of workspace assessment, $\xi = 50°$

Locally calculated L^*		
δ value	Number of iterations	Execution time
0.1	17,509	1.6067
0.07	25,335	2.4502
0.05	35,267	3.1663
0.04	50,919	4.5569
0.025	70,875	6.2210
0.02	102,141	8.8615
0.012	141,881	12.5049
0.009	204,611	20.0849

* The Lipschitz constants are calculated for every box P_i, $i \in \overline{1, k}$

intersection of three rings with inner and outer radii equal to l_u^{min} and l_u^{max} correspondingly, which we plot using dashed lines. The center of every uth ring is moved from the origin in horizontal and vertical direction by $p_u(\xi)$ and $q_u(\xi)$ correspondingly. For the experiments we use the approach with Lipschitzian bounding and locally computed Lipschitz constants.

Table 4 shows the relationship between the δ value and the execution time of the Algorithm 1. As expected the smaller δ we use, the larger number of boxes have to be processed and the better approximation we get.

7 Conclusions

In this paper we proposed an approach to solving systems of nonlinear inequalities with a predefined accuracy based on the nonuniform covering concept. We compared theoretically and experimentally two ways of bounding the constraints on a box: (1) the global optimization with guaranteed accuracy and (2) the extrema approximation based on Lipschitzian bounds. Experimental evaluation suggested that the latter outperforms the former. To demonstrate the practical value of our approach we applied it to the workspace assessment of a 3-DOF planar parallel robot. The obtained results agree with the theoretical expectations.

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